

QUANTUM NEWTON'S LAW

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Abstract

Using the quantum Hamilton-Jacobi equation within the framework of the equivalence postulate, we construct a Lagrangian of a quantum system in one dimension and derive a third order equation of motion representing a first integral of the quantum Newton's law. We then integrate this equation in the free particle case and compare our results to those of Floydian trajectories. Finally, we propose a quantum version of Jacobi's theorem.

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1 Introduction

Recently, Faraggi and Matone derived quantum mechanics from an equivalence postulate which stipulates that all quantum systems can be connected by a coordinate transformation [1, 2, 3]. They deduced that in one dimension, the physical solution of the stationary Schrödinger equation must have the form

$$\phi(x) = \left(\frac{\partial S_0}{\partial x} \right)^{-1/2} \left[\alpha \exp \left(\frac{i}{\hbar} S_0 \right) + \beta \exp \left(-\frac{i}{\hbar} S_0 \right) \right], \quad (1)$$

α and β being complex constants and S_0 the reduced action. In contrast with Bohm's theory, this form of the wave function indicates that the conjugate momentum, defined as

$$P = \frac{\partial S_0}{\partial x}, \quad (2)$$

never has a vanishing value both for bound and unbound states. In the case where the wave function ϕ is real up to a constant phase factor, we have $|\alpha| = |\beta|$ but never $S_0 = cte$. Furthermore, by taking the derivative with respect to x of the expression of S_0 given below, one can see that P is always real even in classically forbidden regions. Within the framework of differential geometry, it is shown in Refs. [1, 2, 3] that the quantum stationary Hamilton-Jacobi equation (QSHJE), leading to the stationary Schrödinger equation in which the wave function is related to S_0 by Eq. (1), is

$$\frac{1}{2m} \left(\frac{\partial S_0}{\partial x} \right)^2 + V(x) - E = \frac{\hbar^2}{4m} \left[\frac{3}{2} \left(\frac{\partial S_0}{\partial x} \right)^{-2} \left(\frac{\partial^2 S_0}{\partial x^2} \right)^2 - \left(\frac{\partial S_0}{\partial x} \right)^{-1} \left(\frac{\partial^3 S_0}{\partial x^3} \right) \right], \quad (3)$$

where $V(x)$ is the potential and E the energy. In Eq. (3), the left hand side reminds us of the classical Hamilton-Jacobi equation and the right one, proportional to \hbar^2 and called the quantum potential, describes the quantum effects. The higher dimension version of Eqs. (1) and (3) is obtained in Ref. [4]. It is also obtained without appealing to differential geometry in Ref. [5]. The solution of Eq. (3), investigated also by Floyd [6, 7, 8, 9] and Faraggi-Matone [1, 2, 3], is given in Ref. [5] as

$$S_0 = \hbar \arctan \left(\frac{\theta_1 + \nu \theta_2}{\mu \theta_1 + \theta_2} \right) + \hbar \lambda, \quad (4)$$

where θ_1 and θ_2 represent two real independent solutions of the Schrödinger equation $-\hbar^2 \phi''/2m + V\phi = E\phi$ and (μ, ν, λ) are real integration constants satisfying the condition $\mu\nu \neq 1$.

Trajectory representation of quantum mechanics, in which the conjugate momentum is different from the mechanical one, was first introduced by Floyd [9, 10] who assumed that trajectories were obtained by using Jacobi's theorem

$$t - t_0 = \frac{\partial S_0}{\partial E}. \quad (5)$$

In classical mechanics, this theorem is a consequence of a particular canonical transformation which is used in Hamilton-Jacobi theory. Let us recall that the classical Hamilton-Jacobi equation is a first order differential equation while the QSHJE is a third order one.

In this letter, we propose a new procedure to determine the motion of any quantum system. In Sec. 2, we construct a Lagrangian from which we derive in Sec. 3 the quantum law of motion. In Sec. 4, we integrate this law in the free particle case and in Sec. 5 we compare our results to Floydian trajectories and propose a quantum version of Jacobi's theorem.

2 Construction of the Lagrangian

First, let us remark that comparing to the usual classical reduced action in one dimension, expression (4) for S_0 contains two additional integration constants μ and ν since S_0 depends also on the constant E through the solutions θ_1 and θ_2 of the Schrödinger equation. This suggests that the fundamental law describing the quantum motion is a differential equation of fourth order since the Newton's classical law is a second order one. This means that the corresponding Lagrangian in the stationary case must be a function of x , \dot{x} , \ddot{x} and maybe of $\ddot{\dot{x}}$ with a linear dependence. However, it is not easy to construct from such a Lagrangian a formalism which leads to the well-known QSHJE. In order to surmount this difficulty, we propose a Lagrangian which is a function of x and \dot{x} , as in classical mechanics, and for which we incorporate two integration constants playing the role of hidden variables. We will later eliminate the indeterminacy introduced by these constants in the formalism by appealing to the QSHJE and its solution. The form of the Lagrangian that we postulate is

$$L(x, \dot{x}, \mu, \nu) = \frac{1}{2}m\dot{x}^2 f(x, \mu, \nu) - V(x) , \quad (6)$$

where $f(x, \mu, \nu)$ is a function which we will determine below. The parameters μ and ν are the non-classical integration constants. As we will see, the function f , and therefore L , depend also on the integration constant E representing the energy of the system.

Now, let us show that the form (6) of the Lagrangian can be justified by appealing to the coordinate transformation, introduced by Faraggi and Matone [3, 11] and called a quantum transformation,

$$x \rightarrow \hat{x} ,$$

after which the QSHJE takes the classical form

$$\frac{1}{2m} \left(\frac{\partial \hat{S}_0(\hat{x})}{\partial \hat{x}} \right)^2 + \hat{V}(\hat{x}) = E . \quad (7)$$

The called quantum coordinate \hat{x} is given by

$$\hat{x} = \int^x \frac{\partial S_0 / \partial x}{\sqrt{2m(E - V(x))}} dx . \quad (8)$$

As shown by Faraggi and Matone [3, 11], setting

$$\hat{S}_0(\hat{x}) = S_0(x), \quad \hat{V}(\hat{x}) = V(x) , \quad (9)$$

Eq. (7) takes the form

$$\frac{1}{2m} \left(\frac{\partial S_0(x)}{\partial x} \right)^2 \left(\frac{\partial x}{\partial \hat{x}} \right)^2 + V(x) = E . \quad (10)$$

The expression of the Hamiltonian can be obtained from (10) with the use of (2)

$$H = \frac{P^2}{2m} \left(\frac{\partial x}{\partial \hat{x}} \right)^2 + V(x) . \quad (11)$$

From (4) and (8), it is clear that \hat{x} is a function depending on x and on the parameters (E, μ, ν) . Therefore, as in classical mechanics, the velocity is given by the canonical equation

$$\dot{x} = \frac{\partial H}{\partial P} = \frac{P}{m} \left(\frac{\partial x}{\partial \hat{x}} \right)^2 . \quad (12)$$

If we set

$$f(x, E, \mu, \nu) = \left(\frac{\partial \hat{x}}{\partial x} \right)^2 , \quad (13)$$

the well-known relation $L = P\dot{x} - H$ leads to the form (6) of the Lagrangian. Using (10), we obtain the following expression for $f(x, E, \mu, \nu)$

$$f(x, E, \mu, \nu) = \frac{1}{2m} \frac{(\partial S_0 / \partial x)^2}{E - V(x)} . \quad (14)$$

If we substitute S_0 with its expression (4), we effectively notice that f depends on x , E , μ and ν . Note that the coordinate \hat{x} is real in classically allowed regions ($E > V$) and purely imaginary in forbidden ones ($E < V$). It follows that the function f which we introduced in the Lagrangian is real positive in classically allowed regions but negative in forbidden ones. This means that in expression (6), the kinetic term in which the well-known quantum potential is absorbed is negative in classically forbidden regions although the velocity \dot{x} is always real.

As observed by Faraggi and Matone [3, 11], when we take the classical limit $\hbar \rightarrow 0$, the quantum coordinate \hat{x} reduces to x . Then, using (13) the function f goes to 1, leading in (6) to the classical form of the Lagrangian. We can therefore consider the Lagrangian (6) as a generalization of the classical one.

Note also that if we define the conjugate momentum from (7) as

$$\hat{P} = \frac{\partial \hat{S}_0}{\partial \hat{x}} = \left(\frac{\partial S_0}{\partial x} \right) \left(\frac{\partial x}{\partial \hat{x}} \right) ,$$

construct the Hamiltonian \hat{H} and then the Lagrangian \hat{L} , we get to the same expression (6) for the Lagrangian by postulating the invariance of L under the quantum transformation. However, in this construction, \hat{P} is not real in classically forbidden regions.

3 The quantum law of motion

Using expression (6) of the Lagrangian, the least action principle leads to

$$mf \ddot{x} + \frac{m}{2} \frac{df}{dt} \dot{x} + \frac{dV}{dx} = 0 , \quad (15)$$

where we have used the fact that $\dot{x}(\partial f / \partial x) = df / dt$. Integrating this last equation gives

$$\frac{1}{2} m \dot{x}^2 f + V = E , \quad (16)$$

which can be shown to be equivalent to (11) if we use (12) and (13). Note that the integration constant E appearing in the right hand side of (16) is already implicitly present in (15) through the function f . Substituting in (16) f by its expression (14), we find

$$\frac{\partial S_0}{\partial x} = \frac{2(E - V)}{\dot{x}}, \quad (17)$$

where we have eliminated one of the roots for $\partial S_0/\partial x$ since Eq. (12) indicates that \dot{x} and $P = \partial S_0/\partial x$ have the same sign in classically allowed regions and are opposite in forbidden ones. Eq. (17), which is a consequence of both the Lagrangian formulation and the QSHJE, will allow us to obtain a fundamental equation describing the quantum motion of any system.

First, note that in the classical case, we have $\partial S_0^{cl}/\partial x = m\dot{x}$ and Eq. (17) reproduces the classical conservation equation

$$E = \frac{1}{2}m\dot{x}^2 + V(x).$$

Let us now derive the quantum conservation equation using the solution (4) of the QSHJE. Setting

$$\phi_1 = \mu\theta_1 + \theta_2, \quad \phi_2 = \theta_1 + \nu\theta_2, \quad (18)$$

we have

$$S_0 = \hbar \arctan\left(\frac{\phi_2}{\phi_1}\right),$$

and then

$$\frac{\partial S_0}{\partial x} = \hbar \frac{\phi_1\phi_2' - \phi_1'\phi_2}{\phi_1^2 + \phi_2^2}. \quad (19)$$

From (17) and (19), we easily deduce

$$\phi_1\phi_2' - \phi_1'\phi_2 = \frac{2}{\hbar} \frac{E - V}{\dot{x}} (\phi_1^2 + \phi_2^2). \quad (20)$$

In order to eliminate the functions ϕ_1 and ϕ_2 and their derivatives, we differentiate this last equation with respect to x . Using the fact that the derivative of the left hand side vanishes since it represents the Wronskian of ϕ_1 and ϕ_2 which are also solutions of the Schrödinger equation, it follows that

$$\phi_1\phi_1' + \phi_2\phi_2' = \frac{1}{2} \left(\frac{1}{E - V} \frac{dV}{dx} + \frac{\ddot{x}}{\dot{x}^2} \right) (\phi_1^2 + \phi_2^2). \quad (21)$$

Now, differentiating (21) with respect to x and using the fact that

$$\phi_1'' = -\frac{2m}{\hbar^2}(E - V)\phi_1, \quad \phi_2'' = -\frac{2m}{\hbar^2}(E - V)\phi_2,$$

we find

$$\begin{aligned} & \left(\phi_1'^2 + \phi_2'^2 \right) - \left(\frac{1}{E - V} \frac{dV}{dx} + \frac{\ddot{x}}{\dot{x}^2} \right) (\phi_1\phi_1' + \phi_2\phi_2') + \left[-\frac{2m}{\hbar^2}(E - V) \right. \\ & \left. - \frac{1}{2} \frac{1}{(E - V)^2} \left(\frac{dV}{dx} \right)^2 - \frac{1}{2(E - V)} \frac{d^2V}{dx^2} - \frac{\dot{x}}{2\dot{x}^3} + \frac{\ddot{x}^2}{\dot{x}^4} \right] (\phi_1^2 + \phi_2^2) = 0. \quad (22) \end{aligned}$$

If we solve the system constituted by (20) and (21) with respect to ϕ'_1 and ϕ'_2 , we can show that

$$\begin{aligned} \phi_1'^2 + \phi_2'^2 = & \left[\frac{4}{\hbar^2} \frac{(E-V)^2}{\dot{x}^2} + \frac{1}{4(E-V)^2} \left(\frac{dV}{dx} \right)^2 \right. \\ & \left. + \frac{\ddot{x}^2}{4\dot{x}^4} + \frac{1}{2(E-V)} \frac{dV}{dx} \frac{\ddot{x}}{\dot{x}^2} \right] (\phi_1^2 + \phi_2^2) . \end{aligned} \quad (23)$$

Now, if we substitute in (22) the quantities $(\phi_1\phi'_1 + \phi_2\phi'_2)$ and $(\phi_1'^2 + \phi_2'^2)$ by their expressions (21) and (23), we find an equation in which all the terms are proportional to $(\phi_1^2 + \phi_2^2)$. We then deduce

$$\begin{aligned} (E-V)^4 - \frac{m\dot{x}^2}{2}(E-V)^3 + \frac{\hbar^2}{8} \left[\frac{3}{2} \left(\frac{\ddot{x}}{\dot{x}} \right)^2 - \frac{\dot{\ddot{x}}}{\dot{x}} \right] (E-V)^2 \\ - \frac{\hbar^2}{8} \left[\dot{x}^2 \frac{d^2V}{dx^2} + \dot{x} \frac{dV}{dx} \right] (E-V) - \frac{3\hbar^2}{16} \left[\dot{x} \frac{dV}{dx} \right]^2 = 0 . \end{aligned} \quad (24)$$

Because it depends on the integration constant E , this equation represents a first integral of the quantum Newton's law (FIQNL). It is a third order differential equation in x containing the first and second derivatives of the classical potential V with respect to x . It follows that the solution $x(t, E, a, b, c)$ of (24) contains four integration constants which can be determined by the initial conditions

$$x(t_0) = x_0 , \quad \dot{x}(t_0) = \dot{x}_0 , \quad \ddot{x}(t_0) = \ddot{x}_0 , \quad \dot{\ddot{x}}(t_0) = \dot{\ddot{x}}_0 . \quad (25)$$

Of course, if we put $\hbar = 0$, Eq. (24) reduces to the well-known first integral of the Newton's classical law $E = m\dot{x}^2/2 + V(x)$. If we solve (24) with respect to $(E-V)$, then differentiate the obtained roots with respect to x , we will obtain the Quantum Newton's Law. It will be a fourth order differential equation in x and will contain the first, second and third derivatives of V , while the classical law $m\ddot{x} = -dV/dx$ is a second order differential equation and contains only the first derivative of $V(x)$.

The second method to derive the FIQNL is to use expression (17) for $\partial S_0/\partial x$ and compute the derivatives

$$\frac{\partial^2 S_0}{\partial x^2} = -\frac{2}{\dot{x}} \frac{dV}{dx} - \frac{2(E-V)\ddot{x}}{\dot{x}^3} , \quad (26)$$

and

$$\frac{\partial^3 S_0}{\partial x^3} = -\frac{2}{\dot{x}} \frac{d^2V}{dx^2} + \frac{6(E-V)\ddot{x}^2}{\dot{x}^5} - \frac{2(E-V)\dot{\ddot{x}}}{\dot{x}^4} + \frac{4\ddot{x}}{\dot{x}^3} \frac{dV}{dx} . \quad (27)$$

Substituting these expressions in the QSHJE given by (3), we obtain the FIQNL, which is exactly the same as that written in (24).

4 The free particle case

Let us examine the case of the free particle for which $V = 0$. The FIQNL takes the form

$$E^2 - \frac{m\dot{x}^2}{2}E + \frac{\hbar^2}{8} \left[\frac{3}{2} \left(\frac{\ddot{x}}{\dot{x}} \right)^2 - \frac{\dot{\ddot{x}}}{\dot{x}} \right] = 0 . \quad (28)$$

In order to solve this differential equation, let us introduce the variables

$$U = \sqrt{2mE} \, x , \quad q = \sqrt{\frac{2E}{m}} \, t , \quad (29)$$

which have respectively the dimensions of an action and a distance. In terms of these new variables, Eq. (28) takes the form

$$\frac{1}{2m} \left(\frac{dU}{dq} \right)^2 - E = \frac{\hbar^2}{4m} \left[\frac{3}{2} \left(\frac{dU}{dq} \right)^{-2} \left(\frac{d^2U}{dq^2} \right)^2 - \left(\frac{dU}{dq} \right)^{-1} \left(\frac{d^3U}{dq^3} \right) \right] . \quad (30)$$

This equation has exactly the same form as (3) when the potential has a vanishing value. This allows us to use the solution (4) to solve Eq. (30). However, if we set

$$\theta_3 = \mu\theta_1 + \theta_2 , \quad (31)$$

the solution (4) takes the form

$$S_0 = \hbar \arctan \left[(1 - \mu\nu) \frac{\theta_1}{\theta_3} + \nu \right] + \hbar\lambda . \quad (32)$$

It follows that the solution of (30) can be written as

$$U = \hbar \arctan \left[a \frac{\psi_1}{\psi_2} + b \right] + U_0 , \quad (33)$$

where a , b and U_0 are real integration constants satisfying the condition $a \neq 0$ and (ψ_1, ψ_2) is a set of two real independent solutions of the equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dq^2} = E\psi .$$

Choosing $\psi_1 = \sin \left(\sqrt{2mE} \, q/\hbar \right)$ and $\psi_2 = \cos \left(\sqrt{2mE} \, q/\hbar \right)$, it follows that

$$x(t) = \frac{\hbar}{\sqrt{2mE}} \arctan \left[a \tan \left(\frac{2Et}{\hbar} \right) + b \right] + x_0 . \quad (34)$$

This relation represents the quantum time equation for the free particle. It is clear that x depends on four integration constants (E, a, b, x_0). In the particular case $a = 1$ and $b = 0$, Eq. (34) reduces to the classical relation

$$x(t) = \sqrt{\frac{2E}{m}} \, t + x_0 .$$

This is compatible with the finding of Floyd [12, 13] who reproduced the classical results by attributing particular values to the non-classical integration constants. However, we do not have the same trajectories. In the next section, we will explain this difference.

5 Floydian trajectories and Jacobi's theorem

In this section, we compare our results with those obtained in Floydian trajectory formulation and propose a quantum version of Jacobi's theorem.

Before going further, we would like to point out that Floyd's conjugate momentum [10] and ours, given in Eq. (12), are both different from the classical one $m\dot{x}$.

Now, let us consider again the free particle case. Using expression (32) as a solution of the QSHJE and choosing $\theta_1 = \sin(\sqrt{2mE} x/\hbar)$ and $\theta_3 = \cos(\sqrt{2mE} x/\hbar)$, we have

$$S_0 = \hbar \arctan \left[a \tan \left(\frac{\sqrt{2mE} x}{\hbar} \right) + b \right] + \hbar \lambda, \quad (35)$$

where we have used the notation $a = 1 - \mu\nu$ and $b = \nu$. Floyd's trajectories are obtained by using Eq. (5)

$$t - t_0 = a \frac{\sqrt{2m/E} x}{(a^2 + b^2 + 1) + \sigma \cos(2\sqrt{2mE} x/\hbar + \gamma)}, \quad (36)$$

where

$$\sigma = \sqrt{a^4 + b^4 + 1 + 2a^2b^2 + 2b^2 - 2a^2}, \quad \gamma = \arctan \left[\frac{2ab}{a^2 - b^2 - 1} \right].$$

First, the relation between t and x contains four integration constants (E , t_0 , a , b) as our result given in (34). We remark also that the classical motion $x = \sqrt{2E/m} (t - t_0)$ is obtained from Floyd's result, Eq. (36), or from ours, Eq. (34), by choosing the particular values $a = 1$ and $b = 0$ for the non-classical integration constants. Note that it is possible to rewrite the trigonometric term appearing in the denominator of Eq. (36) so as to show that this term will vanish for $a = 1$ and $b = 0$. Note also that the values of a and b with which we reproduce the classical results depend on the choice of the two independent solutions of the Schrödinger equation used in the calculation of the reduced action. On the other hand, in the classical limit $\hbar \rightarrow 0$, in both Floyd's approach [9, 12] and ours, a residual indeterminacy subsists except for the particular microstate for which $a = 1$ and $b = 0$. However, by averaging the classical limit of the expression of $t - t_0$ over one cycle of the trigonometric term, Floyd obtained the classical result

$$\langle \lim_{\hbar \rightarrow 0} (t - t_0) \rangle = \sqrt{\frac{m}{2E}} x. \quad (37)$$

Using this procedure, we also obtain the same result after we express t in terms of x in (34). Another interesting question investigated by Floyd concerns microstates [7, 8]. His conclusions, confirmed in Ref. [5], indicate that trajectory representation manifests microstates not detected by the Schrödinger wave function for bound states. Obviously, our approach does not affect these conclusions.

Despite these many common points, it is clear that our trajectories (34) are different from those of Floyd which can be written as in (36). This difference can be explained as follows.

Our first argument concerns the expression of the reduced action as a function of time in the free particle case. In Floyd's approach, it is only in the classical limit $\hbar \rightarrow 0$ that we have

$$\lim_{\hbar \rightarrow 0} S_0 = 2E(t - t_0), \quad (38)$$

while in our approach, from Eq. (17) with $V = 0$, we have

$$dx \frac{\partial S_0}{\partial x} = 2E dt ,$$

leading straightforwardly to

$$S_0 = 2E(t - t_0) , \quad (39)$$

without taking the limit $\hbar \rightarrow 0$. This result is in agreement with the fact that the reduced action is given by

$$S = S_0 - Et = \int_{t_0}^t L dt = \int_{t_0}^t E dt = E(t - t_0) , \quad (40)$$

up to an additive constant term.

Our second argument concerns the use of relation (5). In our point of view, this classical relation resulting in the particular canonical transformation used in the Hamilton-Jacobi theory, must be applied when we use the coordinate \hat{x} with which the QSHJE takes the classical form given in (7). Then, we write

$$t - t_0 = \left[\frac{\partial \hat{S}_0(\hat{x})}{\partial E} \right]_{\hat{x}=cte} . \quad (41)$$

If we substitute in this last equation $\hat{S}_0(\hat{x})$ by $S_0(x)$, we obtain Eq. (5). After this substitution, the derivative with respect to E does not keep \hat{x} invariant as in (41) since Eqs. (13), (14) and (4) indicate that \hat{x} is a function of E . This is the fundamental reason for which our trajectories differ from those of Floyd.

Let us remark that if we take the derivative of Eq. (41) with respect to \hat{x} and then use (7), we obtain

$$\frac{dt}{d\hat{x}} = \frac{\partial}{\partial \hat{x}} \frac{\partial \hat{S}_0(\hat{x})}{\partial E} = \frac{\partial}{\partial E} \frac{\partial \hat{S}_0(\hat{x})}{\partial \hat{x}} = \frac{\partial}{\partial E} \sqrt{2m(E - \hat{V}(\hat{x}))} .$$

Because $\hat{V}(\hat{x}) = V(x)$, we have

$$\frac{dt}{dx} \frac{\partial x}{\partial \hat{x}} = \sqrt{\frac{m}{2(E - V(x))}} . \quad (42)$$

Taking into account the expression of $\partial \hat{x} / \partial x$ which we deduce from (8), Eq. (42) leads to the fundamental relation (17). As explained at the end of Sec. (3), we can derive from (17) the FIQNL without using the Lagrangian formalism. It is therefore possible to obtain the FIQNL, as given in (24), straightforwardly from Jacobi's theorem, as written in (41), and the QSHJE.

We would like to add that Faraggi and Matone [3] also derived an equation which is a first integral of the quantum analogue of Newton's law. This equation depends on the quantum potential and, like ours, is a third order differential equation. However, Faraggi-Matone's derivation is based on relation (5), while our equation can be derived from (41). The two equations are thus different. Concerning Floyd's conjugate momentum [10] and ours, they are also different since the Floyd's is obtained by using (5).

To conclude, we would like to emphasize that the fundamental quantum law of motion (24) was obtained with two different methods from (17), which was itself obtained in two different contexts:

- a Lagrangian formulation by taking advantage of the fact that the solution of the QSHJE is known;
- a quantum version of Jacobi's theorem as written in (41).

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